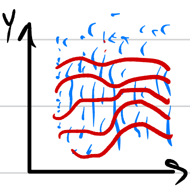
Descriptive Set Theory
Lecture 22

We've shown that 1-1 Borel images of Borel are Borel. How big we can make the preimages of pts s.t. the Borel-ness is still preserved?

Arsenin–Kunugui. K_σ-to-1 Borel images of Borel sets are Borel.

A special case of this is:

Luzin–Novikov. Ctbl-to-1 Borel images of Borel sets are Borel.



In fact, if $R \subseteq X \times Y$ is Borel, X, Y Polish, and each X-fiber is ctbl, i.e. $|R_x| \leq \aleph_0$ $\forall x \in X$, then R is a disjoint ctbl union of Borel function graphs, i.e. $R = \bigcup_n G_n$ $\forall x \forall n \; |(G_n)_x| \leq 1$. In other words, each $G_n = \text{graph}(\gamma_n)$, where $\gamma_n : X \to Y$ is a partial Borel function with a Borel graph. (Since $\text{proj}_X : G_n \to X$ is 1-1, $\text{dom}(\gamma_n) = \text{proj}_X G_n$ is Borel.)

Borel Isom. Theorem. Any two unctbl Polish spaces are Borel isomorphic.

Proof. Enough to show that any unctbl Polish X is Borel isom. to $2^{\mathbb{N}}$. Recall that by Cantor-Bendixson, $2^{\mathbb{N}} \overset{\text{cont.}}{\hookrightarrow} X$. Also, recall the coding lemma: $X \overset{\text{Borel}}{\hookrightarrow} 2^{\mathbb{N}}$. By the Borel Cantor-Schröder-Bernstein theorem, $\exists$ Borel isom. $X \to 2^{\mathbb{N}}$. $\square$

Borel Cantor-Schröder-Bernstein. If A and B are Polish sp. and $\exists$ Borel injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$ then $\exists$ Borel isom. $h : A \to B$.

Proof. Our goal is to obtain Borel partitions $A := \bigsqcup_{n \le \infty} A_n$ and $B := \bigsqcup_{n \le \infty} B_n$ s.t. $\forall n < \infty$, $f(A_n) = B_{n+1}$, $g(B_n) = A_{n+1}$, and $f(A_\infty) = B_\infty$ and $g(B_\infty) = A_\infty$. Because then

he picture:

$$\begin{tikzcd}
A_0 \arrow[dr, Rightarrow, "f"] & B_0 \arrow[dl, Rightarrow, "g"'] \\
A_1 \arrow[dr, Rightarrow, "f"] & B_1 \arrow[dl, Rightarrow, "g"'] \\
A_2 \arrow[dr, Rightarrow, "f"] & B_2 \arrow[dl, Rightarrow, "g"'] \\
A_3 \arrow[dr, Rightarrow, "f"] & B_3 \arrow[dl, Rightarrow, "g"'] \\
\vdots & \vdots \\
A_\infty \arrow[r, Rightarrow, shift left, "f"] & B_\infty \arrow[l, Rightarrow, shift left, "g"]
\end{tikzcd}
\quad \text{obtain} \quad
\begin{tikzcd}
A_0 \arrow[dr, Rightarrow, "f"] & B_0 \\
A_1 \arrow[ur, Rightarrow, "g^{-1}"] & B_1 \\
A_2 \arrow[dr, Rightarrow, "f"] & B_2 \\
A_3 \arrow[ur, Rightarrow, "g^{-1}"] & B_3 \\
\vdots & \vdots \\
A_\infty \arrow[r, Rightarrow, "f"] & B_\infty
\end{tikzcd}$$

define $h : A \to B$

$h|_{A_{2k}} := f|_{A_{2k}}$

$h|_{A_{2k+1}} := g^{-1}|_{A_{2k+1}}$

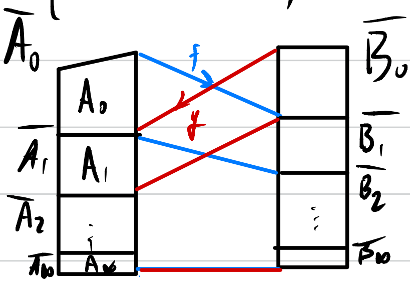
$h|_{A_\infty} := f|_{A_\infty}$,

which is a bijection.

Here, g^{-1} is Borel by Luzin–Souslin. How to obtain such a Borel partition? We'll define decreasing sequences of Borel sets: $\bar{A}_0 := A$, $\bar{B}_0 := B$,

$$\begin{cases} \bar{A}_{n+1} := g(\bar{B}_n) \\ \bar{B}_{n+1} := f(\bar{A}_n) \end{cases} \qquad \bar{A}_\infty := \bigcap_{n<\infty} \bar{A}_n, \quad \bar{B}_\infty := \bigcap_{n<\infty} \bar{B}_n.$$

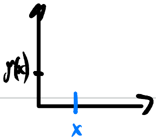
Then we'd let $A_n := \bar{A}_n \setminus \bar{A}_{n+1}$ and $A_\infty := \bar{A}_\infty$, $B_n := \bar{B}_n \setminus \bar{B}_{n+1}$ and $B_\infty := \bar{B}_\infty$. Check using injectivity of f and g that $f(A_n) = B_{n+1}$ and $g(B_n) = A_{n+1}$ and $f(A_\infty) = B_\infty$ and $g(B_\infty) = A_\infty$.



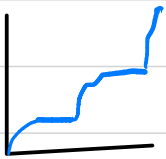
By Luzin–Souslin, all these sets are Borel. $\square$

Measure isom thm. For any nonatomic Borel prob. measure μ on a Polish space X, there is a Borel isom $f : X \to [0,1)$ s.t. $f_* \mu = \lambda$ Lebesgue measure; in particular, (X, μ) and $([0,1), \lambda)$ are isomorphic.

Proof. By the Borel isom theorem, we may assume $X = [0,1)$. Let $f : [0,1] \to [0,1]$ be the function $f(x) := \mu([0,x])$, i.e. f is the distribution function of μ.



Because μ is nonatomic, f is continuous because it's a increasing function and $\lim_{x\to c^-} f(x) = \sum_{x<c} \mu[0,x] = \mu([0,c))$ and $\lim_{x\to c^+} f(x) = \mu(\bigcap_{x>c} \mu[0,x]) = \mu([0,x])$



and since $\mu(\{x\})=0$,
these two are equal. f may not be invertible because it may be constant on some intervals: it is possible that $\mu([a,b]) = 0$ so $f(a) = f(b)$ for some $a<b$. These intervals are cbly many and their union is a μ-null open set U. f is a bijection from $[0,1]\setminus U$ to $[0,1]$. Let $C \subseteq [0,1]$ be the Cantor set and let $C' := f^{-1}(C)$, hence is μ-null. Write $C = C_0 \sqcup C_1$, each C_0, C_1 uncbl. By Borel isom. $\exists$ isom $g_0 : C' \to C_0$ and $g_1 : U \to C_1$. Then the function $h : [0,1] \to [0,1]$ defined by $h|_{[0,1]\setminus(U\cup C')} := f$, $h|_U := g_1$, $h|_{C'} := g_0$.
This h is as desired. $\square$

Based on these isomorphism theorems, we define notions of standard spaces. Namely, a measurable space $(X, \mathcal{B})$ is called standard Borel if $\exists$ Polish top on X s.t. $\mathcal{B}$

is the σ-alg. of Borel sets. We've shown that $\exists!$ up to Borel isomorphism uncountable st. Borel space.

Similarly, we call a probability space $(X, \mathcal{A}, \mu)$ **standard** if $(X, \mathcal{A})$ is a standard Borel space.

We've shown that any nonatomic st. prob. space is isom to $([0,1), \lambda)$.

Examples. (a) For any Polish X, $(X, \mathcal{B}(X))$ is standard Borel by def.

(b) Let X be Polish and let $A \subseteq X$ be a Borel subset. Then the Borel σ-alg. restricted to A is standard. Indeed, we can make A clopen, hence Polish without changing the Borel sets. In fact:

Cor. Let $(X, \mathcal{T}_X)$ be Polish. A subset $B \subseteq X$ is standard Borel (i.e. $\mathcal{B}(X)|_B := \{B \cap A : A \in \mathcal{B}(X)\}$ is standard) if and only if B is Borel.

Proof. $\Leftarrow$. By clopenification as in example (b).

$\Rightarrow$. Suppose $\mathcal{B}(\mathcal{T}_X)|_B$ is standard. Thus,

$\exists$ Polish top τ_B on B s.t. $\mathcal{B}(\tau_B) = \mathcal{B}(\tau_X)|_B$.
This means that the inclusion map $B \hookrightarrow X$ is Borel between Polish space, so it has to map B to a Borel subset of X, i.e. B is Borel. $\square$

Effros Borel structure. Let X be Polish & let $\mathcal{F}(X)$ denote the collection of its closed subsets. The Effros σ-algebra on $\mathcal{F}(X)$ is the one generated by the following sets: let $U \subseteq X$ be open,

$$[U] := \{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}.$$

Thm. For any Polish X, the Effros σ-algebra on X is standard Borel. We call it the Effros Borel space.

Proof. Enough to produce a measurable ison with some Polish space. Define $c : \mathcal{F}(X) \to 2^{\mathbb{N}}$ by fixing a basis (U_n)

$$F \mapsto c(F)$$

where $c(F)(n) := \begin{cases} 1 & \text{if } F \cap U_n \neq \emptyset \\ 0 & \text{o.w.} \end{cases}$. c^{-1} takes the clopen set $[* * \dots * 1]$ to $[U_n]$ and $[* * \dots * 0]$ to $[U_n]^c$, and both are in the Effros σ-algebra, hence c is measurable. This map is also 1-1 by the def of closed.

It's enough to show that $Y := c(\mathbb{F}(X))$ is Borel. In fact we show that Y is G_δ. Indeed, $\forall y \in 2^{\mathbb{N}}$,

$$y \in Y \iff \forall n, m \text{ s.t. } U_m \subseteq U_n \, (y(m) = 1 \Rightarrow y(n) = 1).$$

and

$$\forall n \, (y(n) = 1 \Rightarrow \exists m \, \overline{U_m} \subseteq U_n \; y(m) = 1). \quad \square$$